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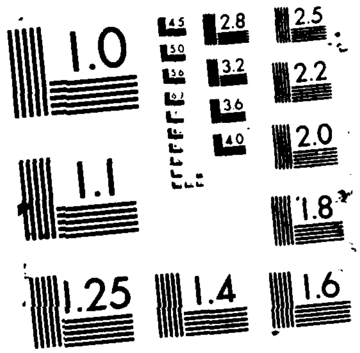
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TYPICAL CLUSTER SIZE FOR 2-DIM PERCOLATION PROCESSES

by

Bao Gia Nguyen

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Abstract: In this paper we discuss the typical cluster size for 2-dim percolation models. We show that, for $W_0 = \{x \in \mathbb{Z}^2 : 0 \to x\}$, $[-\lim_{n\to\infty} \frac{1}{n} P_p(|W_0| = n)]^{-1} \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$ provided that $E_p(|W_0|^2) / E_p(|W_0|) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$. Furthermore, we introduce a new quantity $f_s(p)$, which may be thought of as the singular part of free energy, and show that $f_s(p) \approx |p - p_c|^{d\nu}$ provided that the correlation length $\approx |p - p_c|^{-\nu}$ as $p \to p_c$.

Keywords: Percolation, typical cluster size, singular part of the free energy

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Section 1: Introduction

The purpose of this paper is to discuss some characteristics of the typical cluster size for the self-matching 2-dimensional percolation models. For simplicity ~~we~~ the author only describe ~~our~~ the results for the site percolation model on $\mathbb{Z}^2$ and leave the task of extending ~~our~~ discussion to general models to the readers. Let us now introduce the 2-dim site percolation model. Let P_p denote the probability measure under which all sites of the lattice $\mathbb{Z}^2$ are independently occupied (non-occupied) with probability p (respectively $1-p$). ~~We~~ say that x is connected to y if there is a nearest neighbor path over occupied sites connecting x and y. Let $W_0 = \{x \in \mathbb{Z}^2 : 0 \to x\}$; the cluster of occupied sites connected to 0. ~~Our~~ This paper is devoted to the study of certain special properties of the "typical cluster size" about the critical point $p_c = \inf\{p : P_p(0 \to \infty) > 0\}$. In the paper "Scaling Theory of Percolation Clusters" [1979], Stauffer suggested the following basic postulate: approaches limit of infinity

"We assume that the critical behavior of percolation is dominated by clusters of size $S_\xi \alpha |p - p_c|^{-1/\sigma}$, where differently defined typical cluster size S_ξ all diverge with the same exponent". Furthermore, he also suggested a scaling hypothesis that

$$P_p(|W_0| = n) \alpha \begin{cases} n^{-\zeta+1} \exp(-n/\zeta_\xi(p)) \text{ if } p < p_c & (*) \\ n^{-\zeta+1} \exp(-\sqrt{n/\zeta_\xi(p)}) \text{ if } p > p_c & (**) \end{cases}$$

Nevertheless, it was not clear what Stauffer meant by the "typical cluster size". From (*) we see that

double Z squared : O)
(approaches ... x);

$$E_p(|W_0|^t;\ 0 \not\to \infty)\,\alpha \sum_n n^{(-\zeta+1)+t} \exp(-n/S_\xi(p))$$

hence

$$S_t(p) \equiv E_p(|W_0|^{t+1}; 0 \not\to \infty)/E_p(|W_o|^t;\ 0 \not\to \infty)$$

$$\alpha\ \frac{S_\xi(p)^{-\zeta+3+t}\int_0^\infty x^{-\zeta+2+t}dx}{S_\xi(p)^{-\zeta+2+t}\int_0^\infty x^{-\zeta+1+t}dx}$$

$$\alpha\ S_\xi(p).$$

Similarly we can also see from (**) that $S_t(p)\ \alpha\ S_\xi(p)$. Thus we expect from the scaling theory that for each definition $\xi(p)$ of the correlation length there exists a number $S_\xi(p)$ which decays at the same rate as $S_t(p)$ if $p \to p_c$. We call $S_\xi(p)$ the typical cluster size associated with the correlation length $\xi(p)$. The concept of correlation length is well studied. The usual definitions for the correlation length are

$$\xi(p) = [\inf\{N : P_p(0 \to x) \le \exp(-|x|/N)\}]^{-1} \quad \text{for } p < p_c$$

$$\xi_t(p) = [\sum_x |x|^t P_p(0 \to x;\ 0 \not\to \infty)/\sum_x P_p(0 \to x;\ 0 \not\to \infty)]^{\frac{1}{t}}$$

$$L(p,\varepsilon) = \begin{cases} \min\{n : CR_p(n) \le \varepsilon\} & \text{for } p < p_c \\ \\ \min\{n : CR_p(n) \ge 1-\varepsilon\} & \text{for } p > p_c \end{cases}$$

where $CR_p(n) = P_p$ ($\exists$ an occupied crossing from left to right of the box $B(n)$ of size n centered at 0).

In the definition of $L(p,\varepsilon)$ it is not important to choose a precise value of ε since we can show that for ε smaller than an ε_0, all the above definitions of the correlation length are equivalent in the sense that if $\xi(p) \approx |p-p_c|^{-\nu}$, i.e.

$$\lim_{p\uparrow p_c} \frac{\log\xi(p)}{\log|p-p_c|} = -\nu$$
$$\text{or} \quad p \downarrow p_c$$

then so do the others. From now on we shall fix ε and write $L(p)$ instead of $L(p,\varepsilon)$. For further details on the correlation length we refer the readers to [CCF, 1985], [N1, 1985], or [K3, 1986].

Having introduced the correlation length $L(p)$ we now want to show how to define the typical cluster size $S_L(p)$ associated with $L(p)$. We think of the "typical cluster" as the cluster of all sites in the box $B(L)$ connected to its boundary $\partial B(L)$ by occupied paths and we define

$$S_L(p) = E_p[\#\{x \in B(L) : x \to \partial B(L)\}].$$

This quantity has already been studied extensively by Kesten in [K3, 1986] and was shown to play a very important role in the proofs of scaling relations. As a matter of fact, in that paper Kesten showed that $S_t(p)$ and $S_L(p)$ are equivalent in the sense that $\exists$ constants $A_t, \tilde{A}_t > 0$ so that, for $t > \frac{1}{3}$,

$$A_t S_t(p) \le S_L(p) \le \tilde{A}_t S_t(p).$$

In this paper we take an additional step to observe from (*) that

$$S_I(p) \equiv [\lim_{n\to\infty} -\frac{1}{n}\log P_p(|W_0|=n)]^{-1} \propto S_\xi(p),$$

and then show in section 3 that in fact $S_I(p) \approx S_\xi(p)$ as in the following

<u>Proposition 1</u>: Assume that $S_L(p) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$. Then so does $S_I(p)$; i.e. $S_I(p) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$.

Note that the limit in the definition of $S_I(p)$ exists from the submultiplicative property of $n^{-1}P_p(|W_0| = n)$ (see Kunz-Souillard [1978]). It turns out that the proof of the above result will be based on the following

<u>Lemma</u>: Let $M_t[L(p)]$ = the t^{th} moment of the number of sites in the box $B(L)$ connected to its boundary $\partial B(L)$ by occupied paths, i.e.

$$M_t[L(p)] = E_p[|\{x \in B(L) : x \to \partial B(L)\}|^t].$$

Then

$$M_t[L(p)] \leq B_t[K_1 S_L(p)]^t$$

where $B_t = (t+1)!$ and K_1 is a positive constant depending only on ε.

The proof of the above lemma can be found in [K2, 1986] of Kesten except the fact that $B_t = (t+1)!$. In our opinion it is not easy to see that the B_t's are of order $(t+1)!$ therein since its proof is based on a rather complicated combinatorial argument. Since our proof for proposition 1 depends on the B_t's so we shall give a new proof for the lemma in section 2 with a simple inductive argument.

We now want to note the following

Remarks:

(1) If we apply the estimate in the lemma to the argument in the section 3 of [K3, 1986] we can show that

$$(1.1) \qquad E_p(|W_0|^t;\ |W_0| < \infty) \le C_t\ [K_2 S_L(p)]^t \pi_p(L)$$

where $\pi_p(L) = P_p(0 \to \partial B(L))$ and $C_t = (3t)!$ However, the constants $C_t = (3t)!$ is not strong enough as they were conjectured by Stauffer [1979] that $C_t = t!$ for $p < p_c$, and that $C_t = (2t)!$ for $p > p_c$.

(2) The scaling hypothesis (**) implies that

$$S_{II}(p) \equiv [\lim_{n\to\infty} - \frac{1}{\sqrt{n}} \log P_p(\infty > |W_0| \ge n)]^{-1} \alpha\, S_\xi(p)$$

for $p > p_c$.

We believe in the above but we do not know how to prove this.

Having discussed several ways to look at the typical cluster size we now want to study its role in the singular behavior of the free energy, which is known as the same as the mean number of clusters per site,

$$f(p) = \sum_{n\ge1} \frac{1}{n} P_p(|W_0| = n).$$

It was conjectured in [Sykes-Essam, 1963] that the free energy is singular at p_c. It is not clear at all that the free energy has any singularity since Kesten [K1, 1982] showed that it is twice differentiable. The numerical calculations together with the scaling theory suggested that the third derivative of the free energy should blow up at p_c at the rate $|p - p_c|^{-1-\alpha}$ where the critical exponent α is realted to the exponent ν of the correlation length by the scaling relation (R) $2-\alpha = d\nu$, $d=2$: dimension. Thus we expect that the singular part $f_{sing}(p)$ of the free energy should behave as $|p-p_c|^{d\nu}$ in a neighborhood of p_c. However, it would be difficult to know the singular part since we do not know whether the free energy has any singularity. While it is not easy to define the singular part $f_{sing}(p)$, to prove the scaling relation (R) we propose a new way to look at this. It is based on the observation that if the free energy behaves singularly at p_c then only the tail of the summation in $f(p) = \Sigma_{n\geq 1} n^{-1} P_p(|W_0| = n)$ should play an important role in this singularity. In other words, the mean number of clusters per site should be singular (if it were so!) due to the number of "large clusters". But how large the cluster should be in order for us to see the scaling relationship such as (R)? Physicists (e.g. Stauffer (1979), Essam (1980)) suggested that any cluster which is larger than the typical cluster size should be thought of as the large cluster. From this we believe that

$$f_s(p) \equiv \sum_{n\geq \delta S_L(p)} n^{-1} P_p(|W_0| = n),$$

where δ is some positive constant, should be thought of as a representative for the singular part of the free energy. In order to support our belief, in section 4 we shall apply some recent results of Kesten (see Kesten's theorem in section 4 of our paper) to give an easy proof of the

<u>Proposition 2</u>: Assume that $L(p) \approx |p-p_c|^{-\nu}$ as $p \uparrow p_c$ (or $p \downarrow p_c$).

Then

$$f_s(p) \approx |p - p_c|^{d\nu} \quad \text{as } p \uparrow p_c \text{ (or } p \downarrow p_c)$$

where $d = 2$: the dimension of the percolation model.

Section 2:

Fix ε as in the definition of $L(p,\varepsilon)$. From now on $C_\varepsilon, \tilde{C}_\varepsilon$ will be constants depending only on ε and their value may vary from line to line. Let $\pi_n = P_p$ (0 is connected to a vertical line at distance n away from the origin). It is easy to show

$$(2.1) \qquad \pi_n \asymp P_p(0 \to \partial B(n)) \equiv \pi_p(n)$$

$$(2.2) \qquad \pi_n \asymp \pi_{2n}$$

for all $n \le L(p)$, where $f(p) \asymp g(p)$ means that $\exists\, C_\varepsilon, \tilde{C}_\varepsilon$ such that $\tilde{C}_\varepsilon f(p) \le g(p) \le C_\varepsilon f(p)$.

Recall that $M_t[L(p)] = E_p\{|\{x \in B(L) : x \to \partial B(L)\}|^t\}$ = Average of number of sites connected to the boundary of the box of size $L(p)$. We claim

$$(2.3) \qquad M_{t+1}[L(p)] \le C_\varepsilon (t+1) L(p) [\sum_{k=0}^{2L(p)} \pi_k] M_t[L(p)].$$

To prove this we write

$$M_{t+1}[L(p)] = \sum_{x_1,\ldots,x_{t+1} \in B(L)} P_p(\bigcap_{i=1}^{t+1} \{x_i \to \partial B(L)\})$$

$$= \sum_{x_{t+1}:k=0}^{2L(P)} \sum_{x_1,\ldots,x_t \in B(L)} P_p(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}, x_{t+1} \to \partial B(L))$$

where the index k is the smallest distance from x_{t+1} to the set $\{x_{i=1,\ldots,t}\} \cup \partial B(L)$. For a fixed $k \ge 4$, we have

$$P_p(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}, x_{t+1} \to \partial B(L), \text{Circuit}_{x_{t+1}}(k))$$

$$\le P_p(\bigcap_{i=1}^{t} \{x_i \to \partial B(L) \text{ in } B(L) \backslash B_{x_{t+1}}(k/2)\} \text{ and } \{x_{t+1} \to \partial B_{x_{t+1}}(k/2)\})$$

where $\text{Circuit}_{x_{t+1}}(k)$ is the event that $\exists$ an occupied circuit in the annulus $B_{x_{t+1}}(k)\backslash B_{x_{t+1}}(k/2)$ centered at x_{t+1}. Then by FKG the

$$\text{LHS} \geq P_p(\bigcap_{i=1}^{t+1}\{x_i \to \partial B(L)\})P_p(\text{Circuit}(k)) \geq C_\varepsilon P_p(\bigcap_{i=1}^{t+1}\{x_i \to \partial B(L)\})$$

and

$$\text{RHS} \leq P_p(\bigcap_{i=1}^{t}\{x_i \to \partial B(L) \text{ in } B(L)\backslash B_{x_{t+1}}(k/2)\})P_p\{x_{t+1} \to \partial B_{x_{t+1}}(k/2)\}$$

$$\leq C_\varepsilon P_p(\bigcap_{i=1}^{t}\{x_i \to \partial B(L)\})\pi_k.$$

Hence, for such a $k \geq 4$ we obtain

$$P_p(\bigcap_{i=1}^{t+1}\{x_i \to \partial B(L)\}) \leq C_\varepsilon \pi_k P_p(\bigcap_{i=1}^{t}\{x_i \to \partial B(L)\}).$$

For $k \leq 4$ the above inequality is obvious. Thus we have

$$M_{t+1}[L(p)] \leq \{C_\varepsilon \sum_{k=0}^{2L(p)} 8t(k+1)\pi_k$$

$$+ \tilde{C}_\varepsilon \sum_{k=0}^{L} 8(L-k+1)\pi_p(L-k)\} \sum_{x_1,\ldots,x_t \in B(L)} P_p(\bigcap_{i=1}^{t}\{x_i \to \partial B(L)\})$$

since there are at most $8t(k+1)$ points which are at the distance k from $\{x_1,\ldots, x_t\}$ and there are at most $8(L-k+1)$ points at the distance k from the boundary $\partial B(L)$. Clearly the above shows (2.3).

<u>Remark</u>: In [K2, 1986], Kesten further showed

(2.4) $$\sum_{k=0}^{2L(p)} \pi_k \asymp \sum_{k=0}^{L(p)} \pi_k \asymp L(p)\pi_p(L).$$

Hence, (2.3) implies

(2.5) $$M_{t+1}[L(p)] \leq (t+1)L(p)^2\pi_p(L)M_t[L(p)].$$

Note that

$$M_1[L(p)] = \sum_{x\in B(L)} P_p(x \to \partial B(L)) \le C_\varepsilon \sum_{k=0}^{L(p)} (L-k+1)\pi_k \le K_1 L^2 \pi_p(L).$$

This shows

$$(2.6) \qquad M_{t+1}[L(p)] \le (t+1)!\,[K_1 L^2(p)\pi_p(L)]^{t+1}$$

where K_1 is some positive constant depending only on ε.

Before leaving this section we remark that by the same argument we can show, for $t \ge 1$,

$$(2.7) \quad E_p\{|W_0 \cap B(L)|^t \mid 0 \to \partial B(L)\} \le C_\varepsilon (t+1) L^2 \pi(L) E_p\{|W_0 \cap B(L)|^{t-1} \mid 0 \to \partial B(L)\}$$

and

$$(2.8) \quad E_p\{|W_0 \cap B(L)|^t \mid 0 \to \partial B(L)\} \le (t+1)!\,[K_2 L^2 \pi(L)]^t$$

where K_2 is some positive constant depending only on ε. The inequalities (2.6) and (2.8) play important roles in the proof of (1.1). For a proof of this see [K3, 1986, section 3].

Section 3:

In this section we shall show the proposition 1. First we claim

$$S_I(p) \le C_\varepsilon S_\xi(p). \tag{3.1}$$

The proof given here was suggested to the author by H. Kesten. To prove this it is enough to show $\exists$ $C_1, C_2, C_3 > 0$ so that

$$P_p(|W_0| \ge C_1 k L^2 \pi(L)) \le C_2 \exp(-C_3 k). \tag{3.2}$$

We denote $B(\underline{n})$ the boxes of size $L(p)$ centered at $\underline{n} = (n_1 2L, n_2 2L)$, $(n_1, n_2) \in \mathbb{Z}^2$. We say that $\underline{n}$ is connected to $\underline{0}$ if $\exists$ an occupied path connecting the $B(\underline{n})$ and $B(\underline{0})$. Let $C = \{\underline{n} : 0 \to \underline{n}\}$. It can be seen from the proof of the theorem 5.1 of [Kesten, 1982] that

$$P_p(|C| > k) \le \tilde{C}_2 \exp(-\tilde{C}_3 k) \tag{3.3}$$

for some positive constants $\tilde{C}_2, \tilde{C}_3$. Thus to show (3.2) it is enough to show the exponential decay of $P_p(|W_0| \ge C_1 k L^2 \pi(L); |C| \le k)$. Note that the number of clusters C with $|C| \le k$ is bounded by C_4^k for some positive constant C_4. Fix such a cluster $C = \{\underline{n}_1, \ldots, \underline{n}_\ell\}$; $\ell \le k$. Let

$$X_{\underline{n}_i} = |\{x \in B(\underline{n}_i) : x \to \partial B(\underline{n}_i)\}|.$$

We have

$$\begin{aligned} P_p(|W_0| \ge C_1 k L^2 \pi_p(L); C) &\le P_p\Big(\sum_{\underline{n}_i \in C} X_{\underline{n}_i} \ge C_1 k L^2 \pi_p(L); C\Big) \\ &\le \inf_{r>0} e^{-rC_1 k L^2 \pi_p(L)} E_p \exp\{(\sum_{\underline{n} \in C} r X_{\underline{n}_i}); C\} \\ &\le \inf_{r>0} e^{-rC_1 k L^2 \pi_p(L)} [E_p(e^{rX_{\underline{n}_1}})]^\ell \end{aligned}$$

But

$$E_p(e^{rX_{\underline{n}_1}}) = \sum_{t=0}^{\infty} \frac{r^t}{t!} E_p(X^t_{\underline{n}_1}) \le \sum_{t=0}^{\infty} \frac{r^t}{t!}(t+1)!\,[K_1L^2r_p(L)]^t = \sum_{t=0}^{\infty} r^t(t+1)\,[K_1L^2\pi_p(L)]^t.$$

Now we choose $r = 1/2K_1L^2\pi_p(L)$. Then

$$E_p(e^{rX_{\underline{n}_1}}) \le \sum_{t=0}^{\infty} (t+1)\left(\tfrac{1}{2}\right)^t \le C_5 < \infty \;.$$

Thus

$$P_p(|W_0| \ge C_1kL^2\pi_p(L);\, |C| \le k) \quad C_4^k e^{-\frac{C_1}{K_1}k}(C_5)^k.$$

Choose $C_1 = 2K_1x_0$, where $x_0 = \log C_4C_5$, to obtain (3.2). Thus from (3.1) the critical exponent of $S_I(p)$ is not larger than Δ. To get the other bound we consider

$$E_p(|W_0|^t) = \sum_{n=1}^{\infty} n^tP_p(|W_0| = n) \le \sum_{n=1}^{\infty} n^t n\exp(-n|S_I(p)) \le KS_I(p)^{t+2}$$

where K is some positive constant. But in [K3; 1986] Kesten showed that

$$E_p(|W_0|^t) \ge C_tS_L(p)^t\pi_p(L)$$

where C_t is some constant depending on t. Then

$$C_tS_L(p)^t\pi_p(L) \le KS_I(p)^{t+2}.$$

Hence,

$$-\frac{\log S_L(p)}{\log|p-p_c|} - \frac{1}{t}\,\frac{\log C_t\pi_p(L)}{\log|p-p_c|} \le -\frac{t}{t+2}\,\frac{\log KS_I(p)}{\log|p-p_c|}\;.$$

Letting $p \uparrow p_c$ and then $t \to \infty$, we obtain the result that $S_I(p) \approx |p-p_c|^{-\Delta}$.

Section 4:

The proof of proposition 2 will be based on the following results in [K3, 1986].

Theorem (Kesten):

Let $L(p)$ and $\pi_p(L(p))$ as before. Then we have

(a) $E_p(|W_0|; |W_0| < \infty) \asymp \pi_p^2(L)L^2(p)$

(b) $S_L(p) \asymp \pi_p(L)L^2(p)$

(c) $\exists$ a positive constant δ such that

$$P_p(\infty > |W_0| \geq \delta S_L(p)) \geq \frac{1}{2}\pi_p(L).$$

We omit the proof of this theorem and refer the reader to find its proof in the combination of the two papers [K2 and K3, 1986]. Once the theorem is established the rest will be easy. In fact on one hand we have from the Cauchy-Schwartz inequality that

$$\Big[\sum_{n \geq \delta S_L(p)} nP_p(|W_0| = n)\Big]\Big[\sum_{n \geq \delta S_L(p)} \frac{1}{n}P_p(|W_0| = n)\Big]$$
$$\geq \Big[\sum_{n \geq \delta S_L(p)} P_p(|W_0| = n)\Big]^2 \geq \Big[\frac{1}{2}\pi_p(L)\Big]^2$$

by (c) of the above theorem. Thus

$$\begin{aligned} f_s(p) \equiv \sum_{n \geq \delta S_L(p)} n^{-1}P_p(|W_0| = n) &\geq \frac{1}{4}\pi_p^2(L) \Big/ \sum_{n \geq \delta S_L(p)} nP_p(|W_0| = n) \\ &\geq \frac{1}{4}\pi_p^2(L)/E_p(|W_0|; |W_0| < \infty) \\ &\geq \frac{1}{4}\pi_p^2(L)/C_\varepsilon \pi_p^2(L)L^2(p) \\ &\geq \frac{1}{4C_\varepsilon L^2(p)} \end{aligned}$$

where in the last inequality we used (a). On the other hand,

$$f_s(p) \le \frac{1}{[\delta S_L(p)]^2} \sum_{n \ge \delta S_L(P)} nP_p(|W_0| = n)$$

$$\le \frac{1}{\delta^2 S_L^2(p)} E_p(|W_0|; |W_0| < \infty)$$

$$\le \frac{C_\varepsilon L^2(p)\pi_p^2(L)}{\tilde{C}_\varepsilon \delta^2 [L^2(p)\pi_p(L)]^2} = \frac{C_\varepsilon}{\tilde{C}_\varepsilon \delta^2 L^2(p)}$$

by (b). Since $L(p) \approx |p - p_c|^{-\nu}$ we obtain the proposition.

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